The Fibonacci Project

DISSEMINATING INQUIRY-BASED SCIENCE AND MATHEMATICS EDUCATION IN EUROPE

Background Resources For Implementing Inquiry in Science and Mathematics at School

Inquiry in Mathematics Education

WITH THE SUPPORT OF

EUROPEAN COMMISSION

SEVENTH FRAMEWORK PROGRAMME

Resources for Implementing Inquiry in Science and Mathematics at School

The Fibonacci Project (2010-2013) aimed at a large dissemination of inquiry-based science education and inquiry-based mathematics education throughout the European Union. The project partners created and trialled a common approach to inquiry-based teaching and learning in science and mathematics and a dissemination process involving 12 Reference Centres and 24 Twin Centres throughout Europe which took account of local contexts.

This booklet is part of the *Resources for Implementing Inquiry in Science and in Mathematics at School*. These Resources include two sets of complementary booklets developed during the Fibonacci Project:

1) Background Resources

The *Background Resources* were written by the members of the Fibonacci Scientific Committee. They define the general principles of inquiry-based science education and inquiry-based mathematics education and of their implementation. They include the following booklets:

- 1.1 Learning through Inquiry
- 1.2 Inquiry in Science Education
- 1.3 Inquiry in Mathematics Education

2) Companion Resources

The *Companion Resources* provide practical information, instructional ideas and activities, and assessment tools for the effective implementation of an inquiry-based approach in science and mathematics at school. They are based on the three-year experiences of five groups of Fibonacci partners who focused on different aspects of implementation. The *Companion Resources* summarise the lessons learned in the process and, where relevant, provide a number of recommendations for the different actors concerned with science and mathematics education (teachers, teacher educators, school directives, deciders, policy makers...). They include the following booklets:

- 2.1 Tools for Enhancing Inquiry in Science Education
- 2.2 Implementing Inquiry in Mathematics Education
- 2.3 Setting up, Developing and Expanding a Centre for Science and/or Mathematics Education
- 2.4 Integrating Science Inquiry across the Curriculum
- 2.5 Implementing Inquiry beyond the School

Reference may be made within this booklet to the other *Resource* booklets. All the booklets are available, free of charge, on the Fibonacci website, within the *Resources* section.

Editorial coordinator : Susana Borda Carulla

Fibonacci Scientific Committe : Michèle Artigue, Peter Baptist, Justin Dillon, Wynne Harlen, David Jasmin, Pierre Léna

WWW.FIBONACCI-PROJECT.EU

Fibonacci Project, December 2012

This project has received funding from the European Union's Seventh Framework Programme

EUROPEAN COMMISSION

SEVENTH FRAMEWORK PROGRAMME

Inquiry in Mathematics Education

Michèle ARTIGUE, Peter BAPTIST

Table of contents

Introduction

In recent decades, inquiry-based education and inquiry-based learning, whose roots can be traced at least to Dewey's approach towards education nearly one century ago (Dewey, 1926, 1933, 1938), have been gaining influence in education, but this terminology has remained essentially attached to science education. The authors of the report known as Rocard's report (Rocard & al., 2007), which has been very influential in promoting inquiry-based science education (IBSE) at the European level, seem themselves hesitant to extend this terminology to mathematics. They write:

> *"In mathematics teaching, the education community often refers to "Problem-Based Learning" (PBL) rather than to IBSE. In fact, mathematics education may easily use a problem based approach while, in many cases, the use of experiments is more difficult. Problem-Based Learning describes a learning environment where problems drive the learning. That is, learning begins with a problem to be solved, and the problem is posed in such a way that children need to gain new knowledge before they can solve the problem. Rather than seeking a single correct answer, children interpret the problem, gather needed information, identify possible solutions, and evaluate options and present conclusions." (p.9)*

In the Fibonacci Project, the same terminology was used both for mathematics and sciences, hence the expression 'inquiry-based mathematics education' (IBME). This makes it necessary to clarify exactly how we understand this expression, and how its characteristics compare with those of IBSE. The first part of this document, written by Michèle Artigue, is devoted to such clarifications, while in the second part Peter Baptist focuses on the implementation of IBME, on the important changes that the move from traditional education to IBME requires, and on how these changes can be progressively achieved. The view of IBME expressed in this booklet is coherent with the nine key features of inquiry pedagogy (basic patterns) that emerged from the modules of the German SINUS project (1998-2007) and were adopted by the Fibonacci project, and which are described in detail in the Fibonacci Background Resource Booklet *Learning Through Inquiry*[1].

1. What is Inquiry-Based Mathematics Education (IBME)?

Michèle ARTIGUE
Université Paris Diderot, France

In the introduction we pointed out that, contrary to what occurs in science education, the term inquiry-based mathematics education (IBME) is of recent use in mathematics education. Nevertheless, several attempts at promoting teaching practises with evident similarities to those associated with inquiry-based science education (IBSE) can be found in this field. Indeed, throughout the last fifty years, one of the main ambitions of innovation and research in the field of mathematics education has been to promote mathematical learning with understanding, to help pupils experience authentic mathematical activity from the early grades, and to elucidate the conditions for making this possible. It is thus necessary to consider IBME in relation to this heritage, and investigate how the scientific resources it provides can be used for developing and strengthening IBME[2].

The Fibonacci Background Resource Booklet *Inquiry in Science Education*[3] points out that "inquiry is a term used both within education and in daily life to refer to seeking knowledge or information by asking questions", and that "what distinguishes scientific inquiry is that it leads to knowledge and understanding of the natural and

1 Available at www.fibonacci-project.eu, in the *Resources* section.
2 A more substantial reflection on this issue is developed in Artigue & Blomøj (in press).
3 Available at www.fibonacci-project.eu, in the *Resources* section.

made word around us". The text also proposes a model of learning science through inquiry which is summarised in the schema reproduced below (fig. 1) and described as "the process of building understanding through collecting evidence to test possible explanations and the ideas behind them in a scientific manner" (p. 9).

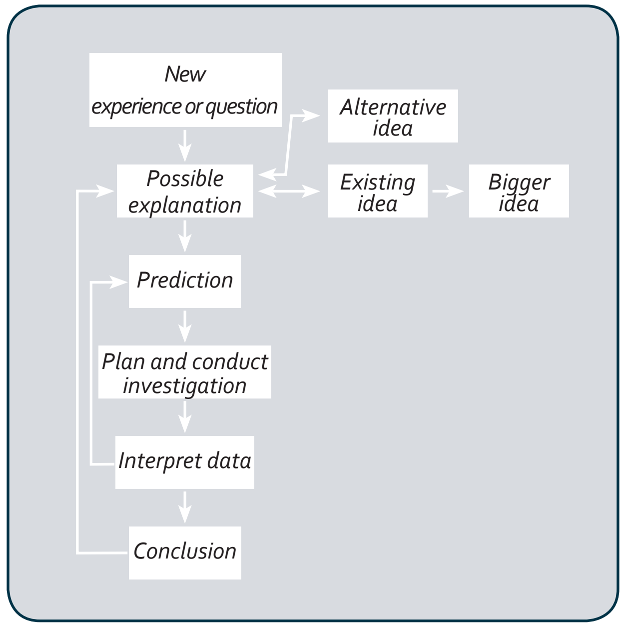


Fig. 1: A model of learning science through inquiry.

As pointed out in the Fibonacci Background Resource Booklet *Learning Through Inquiry*[4], mathematical inquiry presents evident similarities with scientific inquiry as described above. Like scientific inquiry, mathematical inquiry starts from a question or a problem, and answers are sought through observation and exploration; mental, material or virtual experiments are conducted; connections are made to questions offering interesting similarities with the one in hand and already answered; known mathematical techniques are brought into play and adapted when necessary. This inquiry process is led by, or leads to, hypothetical answers – often called conjectures – that are subject to validation.

This is rarely a linear process. Quite often, initial conjectures are found to be true only under specific conditions, which may lead to their modification, or even to questioning the definition of the mathematical objects involved (as illustrated by Lakatos (1975) in his paradigmatic study on Euler's formula for polyhedrons). Further, the process may lead to new questions and problems whose solution may affect the answers to the initial question, or even the formulation of the question itself.

Nevertheless, despite the existence of similarities with scientific inquiry, mathematical inquiry has some distinct specificities, both regarding the type of questions it addresses and the processes it relies on to answer them.

1.1 Mathematical inquiry: internal and external questions

As in scientific inquiry, mathematical inquiry is often motivated by questions arising from the natural world or the made world around us. But if one main ambition of mathematics is to contribute to understanding of the natural, social and cultural world, and to empower human beings to act on this world, it should not be forgotten

4 Available at www.fibonacci-project.eu, in the *Resources* section.

that mathematics as a science also creates its own objects and reality, and that the questions raised by these objects have always been an essential motor of its development. As stressed in the Fibonacci Background Resource Booklet *Learning Through Inquiry*:

> *"As they become familiar, mathematical objects also become the terrain for mathematics experimentation. Numbers, for instance, have been used for centuries and are still an incredible context for mathematics experiments, and the same can be said of geometrical forms. Patterns play a great role in mathematics, whether they are suggested by the natural world or fully imagined by the mathematician's mind. Digital technologies also offer new and powerful tools for supporting investigation and experimentation in these mathematical domains. IBME must, therefore, not just rely on situations and questions arising from real world phenomena, even if the consideration of these is of course very important, but use the diversity of contexts which can nurture investigative practices in mathematics."*

Thus the sources of mathematical inquiry in IBME and the associated questions may be very diverse. They can emerge from:

- natural phenomena (e.g.: how to understand and characterise changes in the shadow of an object cast by the sun?),
- technical problems (e.g.: how to measure inaccessible magnitudes and objects?),
- human artefacts (e.g.: what is the effect of a pantograph on geometrical figures and why? How does a GPS work?),
- art (e.g. what are the symmetries of an architectural object or piece of art? What are the minimal elements which can be used for generating a periodic tessellation?),
- daily life problems (e.g. how to choose between different offers for mobile telephony and internet?).

But mathematical objects themselves from the early ages can be an essential source of mathematical inquiry.

- What is the greatest product that can be obtained by decomposing a positive integer into a sum of positive integers and multiplying the terms of the sum?
- Can all positive integers be obtained as the difference of two squares of integers? Are all positive integers the sum of consecutive positive integers?
- What can it mean for two triangles, two rectangles, two polygons to have the same form?
- Given two triangles with the same area, can they be transformed one into the other by cutting and pasting? Does this extend to any pair of polygons?
- If two triangles have the same perimeter and the same area, are they necessarily isometric?

The nature of the question obviously has an impact on the inquiry process. In the case of questions from an external source, such as in the first examples mentioned above, transforming these questions into questions accessible to mathematical work is an important part of the process of inquiry, engaging a modelling process[5]. In recent decades, research in mathematics education has paid more and more attention to these processes as shown, for instance, by the activities of the international group ICTMA or the extended literature devoted to this theme[6]. This literature generally presents modelling as a cyclic process, which creates some similarity with the model for IBSE in figure 1, at least at a surface level.

5 Models used in science are not necessarily mathematical models, and even when this is the case, models do not necessarily take the form of laws and equations as is mostly the case in physics. They can, for instance, be geometrical shapes (DNA, fullerenes, proteins), graphs or symbolic codes (as in chemistry and genetics), but internal logical consistency and the ability to go beyond a simple heuristic description remain the absolute requirements for any good model. In mathematics education and IBME, the term modelling is thus used in a restricted sense: it refers to a process engaging mathematisation and the construction of mathematical models.

6 See the activities of the international group ICTMA (www.**ictma**.net) or (Kaiser & al., 2011).

Figure 2 presents one classical vision of this modelling cycle reproduced (from Blomhøj & Højgaard Jensen, 2003, p.127) with the corresponding description of the different phases:

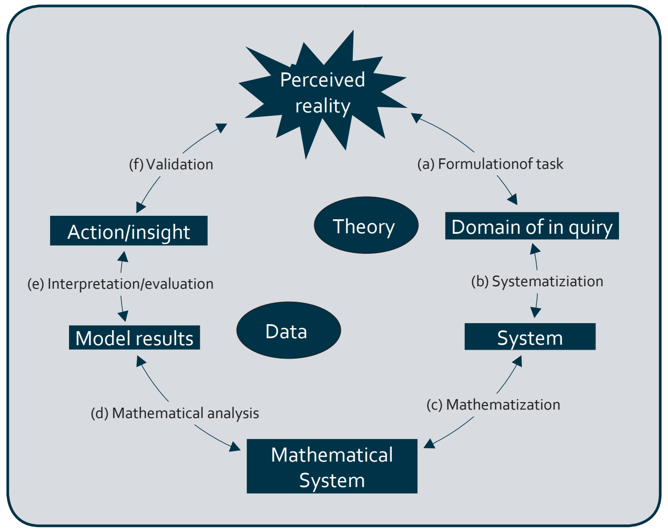


Figure 2 : Modelling cycle

> *"a) Formulation of a task (more or less explicit) that is related to a perceived reality and influenced by the modeller's interests. Through this process the object of the modelling process is constructed. The object can be reconstructed as a result of the modelling process. However, it is the object and the formulated task that guide the identification and construction of a domain of inquiry.*
>
> *b) Selection and construction of the relevant objects, relations etc. from the domain of inquiry, and idealisation of these, in order to make a mathematical representation possible.*
>
> *c) Transformation and translation of selected objects and relations from their initial mode of appearance to mathematics by further abstraction and idealisation.*
>
> *d) Using mathematical methods to achieve mathematical results and conclusions.*
>
> *e) Interpretation of these as results and conclusions regarding the system or the initiating domain of inquiry.*
>
> *f) Evaluating the validity of the model by comparison with data (observed or predicted) and/or with already established knowledge (theoretically based or shared/personal experience based)."*

This modelling cycle thus organises the relationship and interaction between two systems: an extra-mathematical system and a mathematical system. Each obeys its own logic, and thus the inquiry process is not only subject to the rules of mathematical rationality. As is made clear by the description above, mathematical rationality regulates the work carried out in phase d, and this work can itself have an inquiry dimension, but validation of the answers obtained through the inquiry process as a whole is also submitted to the rationality of the extra-mathematical system. In IBME, it is certainly important to make students aware of this and give them the opportunity to experience the diversity of domains that are accessible to mathematical inquiry through modelling processes, beyond the sole natural world.

Comparing with IBSE, beyond the fact that the modelling process does not necessarily use mathematics, it should be noted that the loop may often involve more preconceived ideas than in mathematics, since the natural world is more immediately perceived by the senses, and requires a crude model to be dealt with, even at early ages. These pre-conceptions may have a strong impact on the process.

It should be added that inquiries motivated by mathematical questions (internal inquiries) may also require or benefit from building interactions between different systems (between numerical and algebraic systems, algebraic and geometrical systems, deterministic and stochastic perspectives, etc.) since mathematics is a highly connected field. Such interactions can be seen as specific forms of modelling processes, internal to mathematics[7].

1.2 Mathematical inquiry: some specificities of the process

Box 1 presents an example of an inquiry process based on a mathematical question: What is the greatest product that can be obtained by decomposing a positive integer into a sum of positive integers and multiplying the terms of the sum?

This question has led to the development of many different activities at different levels of schooling, from primary education to senior high school (ERMEL, 1999; Aldon & al., 2010).

Box 1. The greatest product

What is the greatest product that can be obtained by decomposing a positive integer into a sum of positive integers and multiplying the terms of the sum?

Let's select for instance the number 10 and carry out some exploration.

If 10 is decomposed into the sum of two numbers, an exhaustive exploration of possibilities is straightforward. We might quickly conclude that the greatest product is obtained through the balanced decomposition 10=5+5, leading to the product 5x5=25. This conjecture is often expressed at an early stage of the inquiry process, before any systematic work has been engaged.

Nevertheless, the 5+5 conjecture does not survive when 10 is decomposed into the sum of three numbers (for instance 10=5+3+2, leading to the product 5x3x2=30). In that case, systematic exploration requires consideration of many different cases. Yet that does not solve the problem, since 10 can also be decomposed into more than three numbers.

Thus, we engage in a dialectic process involving trials and the progressive elaboration of partial results and conjectures. For instance, quite soon we find that optimal decompositions should not include the number 1 nor the number 5 (because 3x2>5), nor the number 6 (because 3x3>6). This line of reasoning excludes all numbers different from 2, 3 and 4. We then notice that 3x3>2x2x2, and that in any decomposition the number 4 can be replaced by two number 2 (as in 4=2+2=2x2). Thus, we finally obtain two optimal decompositions: 10=3+3+4 and 10=3+3+2+2 both leading to the product 36.

In fact, the work carried out in this particular case applies to the case of any whole number, optimal decompositions being those including as many of number of 3 as possible, and no number 1. It can also generalise to rational decompositions as shown for instance by Perrin (ERMEL, 1999). In that case, if n is the number of terms of the decomposition of number *S*, convexity properties lead to the fact that the optimal decomposition is the balanced decomposition in *n* terms of value *S/n*. Thus one is led to seek the maximum of the function of $n{:}(S/n)^n$. Extending this function to real numbers, and calculating the derivative, one finds that the maximum of this real function is obtained for $x=e$, and the optimal decomposition in rational numbers is thus obtained by choosing *n* such that *S/n* is as close as possible to *e*. This gives new insight into the result obtained for 10 and then generalised to all decompositions with whole numbers. 3 is the integer closest to *e*!

7 Douady has put specific emphasis on this point in her approach to mathematics education in terms of tool-object dialectics and interplay between settings (Douady, 1986).

This problem is just one example but it shows some specificities of (internal) mathematical inquiry and of its outcomes, for instance:

- the role played by exploration and its progressive organisation as familiarity with the problem increases;
- the pragmatism of the inquiry process and its non linearity;
- the dialectic interplay between proof and refutation, and the role played in it by counter-examples;
- the definitive (apodictic) nature of the results obtained and the conviction that no further experience will invalidate them, but also the intellectual satisfaction that one gets when discovering new reasons for results already proved;
- the fact that once a solution is found, one immediately looks for possible generalisations, considering both the results and the techniques used for obtaining them;
- the change in vision that such generalisations may require, calling on new mathematical domains and techniques, and how they can contribute to new understanding of the initial results obtained.

This being said, there is no unique or standard form of inquiry in mathematics, even for forms of inquiry internal to the mathematical field as in the example in Box 1. Such an inquiry process, for instance, can lead to a universal result (showing that all objects belonging to the same category share a given property), but it can also lead to an existential result (showing that there exists at least one object fulfilling a set of given conditions). The development of inquiry and the validation process it includes will necessarily be affected by these characteristics, as it will be affected by the nature of the objects involved, the students' conceptualisation of them, the language, symbolic and semiotic tools accessible to them for expressing and discussing their ideas and reasoning with other students and the teacher, and the artefacts accessible for supporting exploration and experimentation. Such diversity together with the fact that mathematical inquiry, when motivated by external issues, necessarily includes a modelling process and combines several logics, suggests that in practise there is a continuum between the mathematical and scientific forms of inquiry that inquiry-based education can involve, as pointed out for instance in Hersant & Orange-Ravachol (2012).

1.3 The connected and cumulative dimension of mathematics

The booklet *Learning Through Inquiry* states that:

> *"Mathematics has a cumulative dimension to a greater extent than science. Mathematical tools developed for solving particular problems need to build on each other to become methods and techniques which can be productively used for solving classes of problems, eventually leading to new mathematical ideas and even theories, and new fields of application. Moreover, connections between domains play a fundamental role in the development of mathematics. Thus it is important in implementing inquiry-based mathematics education that students not deal only with isolated problems, however challenging they may be, since this may not enable them to develop the over-arching (or more generally applicable) mathematical concepts." (p. 9)*

This suggests that IBME should pay special attention to the development of resources starting from problems or questions whose potential for progressing in mathematical ideas and knowledge is clear. Although the development of "inquiry habits of minds" (Dewey, 1938) are an important dimension of IBME, it is clearly not sufficient.

From this perspective, despite its interest, a problem such as that of the greatest product (Box 1) presents some limitations, as the crucial role it can play in a given progression of mathematical knowledge is not self-evident. The case is different for instance with the well known situation of the puzzle, initially created by Brousseau (Brousseau & Brousseau, 1987) and described in Box 2.

Box 2. The puzzle situation

This situation is part of a didactical engineering process whose aim is to extend the field of numbers from whole numbers to rational and decimal numbers. At this point of the engineering process, students have been introduced to fractions through comparing and evaluating the thickness of different types of paper. They know how to compare and add fractions, and thus how to multiply a fraction by an integer. But they do not yet have the tools to find the product of two fractions. Within the didactical engineering process, these tools are built by presenting multiplication as a linear operator in the context of enlargement of objects.

The puzzle situation is the first associated activity to provide students which such tools. Students are asked to enlarge the puzzle shown in figure 3, in such a way that the length of 4cm in the initial puzzle becomes 7cm in the enlarged puzzle (the initial side of the square is 11cm).

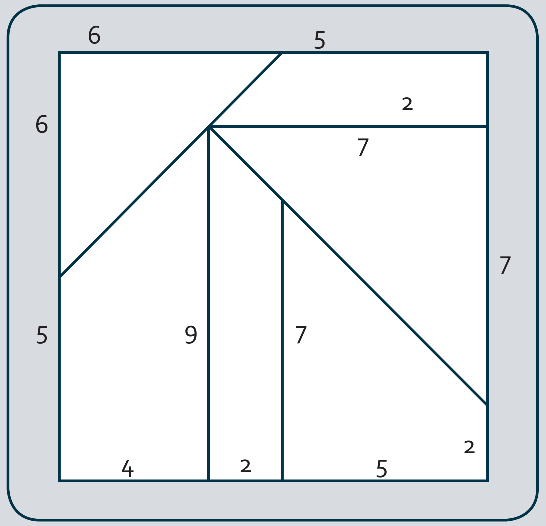


Figure 3: Brousseau's puzzle

They work in groups and each group is asked to produce an enlarged puzzle. Work is distributed between the students, each of them having to enlarge one or two pieces of the puzzle. As there is no simple ratio between 4 and 7, their initial strategy is to add 3cm to all dimensions (additive model). After checking that the strategy has been correctly applied for obtaining the enlarged pieces, they realise that it does not work, and thus look for another strategy. With the support of the teacher, this progressively leads them to the idea that if they know the length of the image of the 4cm segment, they can calculate the length of the image of the 2cm segment and that of the 6cm segment. They eventually understand that, in more general terms, knowing length of the image of a 1cm segment allows them to calculate the image of a segment of any length, and that this technique can be generalised to any similar situation. Later on, they will be asked to extend this technique to work with rational dimensions, which will complete the desired extension of multiplication.

The puzzle situation described in Box 2 deserves some comment. It shows that in mathematics, just as in science, knowledge progression is not just a linear process. Models such as the additive model, which have proved their efficiency in a number of contexts, must at some point be questioned by the students and new constructions developed. Epistemological obstacles exist that need to be overcome (Sierpinska, 1996). Within IBME, the teacher must lead students to experience the limitations of their knowledge and create the conditions for achieving the required cognitive evolution.

Beyond the design of appropriate problems, such learning experiences require appropriate conditions for students' interaction with the problem. These conditions are captured by the ideas of *situation* and *milieu* in Brousseau's theory of didactical situations (Brousseau, 1997). In the situation described in Box 2, students are physically confronted with the limitations of the additive model, which paves the way for the required cognitive evolution. Understanding why the additive model does not work in this particular situation leads to the essential idea that the image of the sum of two lengths should be equal to the sum of their images, the image of 4cm should be twice that of 2cm, that is to say, to the essence of the linear model. Nevertheless, as Sensevy (2011) points out, the joint dialogic action of the teacher and the students, and the accurate use of semiotic resources such as tables with measures, is crucial in order for this potential evolution to become a reality and for the inquiry process to lead to the expected conclusion. The "key features of inquiry pedagogy", described in the booklet *Learning Through Inquiry*, pay particular attention to this essential dialogic dimension of the learning process in IBME.

1.4 IBME and digital technologies

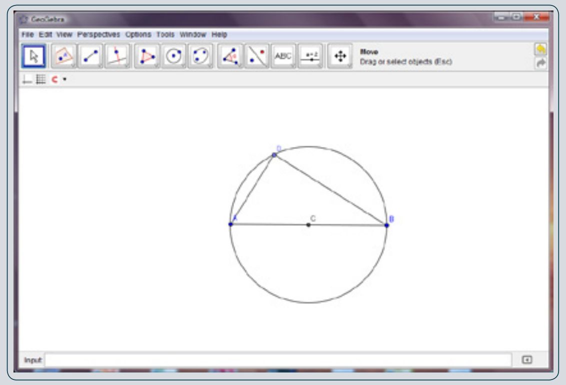
Quoting again from *Learning Through Inquiry*:

> *"If it is to be more than a slogan, IBME requires the development of appropriate educational strategies. These strategies must acknowledge the experimental dimension of mathematics and the new opportunities that digital technologies offer in support. The history of mathematics shows that such an experimental dimension is not new, but over the last decades technological developments have put a large number of new resources at the disposal of mathematics which have made the experimental dimension more visible and shared by the mathematical community. Compared with experimental practises in natural science, however, one must keep in mind that the terrain of experience for mathematics learning is not limited to what is usually called the "real world"." (p. 8)*

As pointed out here, mathematics has always had an experimental dimension. The educational importance of this experimental dimension has been stressed by mathematicians such as Felix Klein in Germany or Emile Borel in France who, at the beginning of the 20th century, pleaded for the installation of mathematics laboratories in French high schools. The situations described in Box 1 and Box 2 show that, even today, experimentation in mathematics is not dependent on the use of digital technologies. This being said, there is no doubt that technological developments have radically enhanced the potential for experimentation in mathematics and in mathematics education and that IBME must make use of these new technological resources. It is crucial to keep in mind, however, that the use of technology does not automatically reinforce the inquiry dimension of teaching and learning practises. This point is illustrated by the example in Box 3.

Box 3. Triangle and circle – the right angle theorem

Students work with an interactive geometry application such as the one captured in the screen shot below. D is a mobile point on the perimeter of a circle of diameter [AB]. Students are asked to make conjectures about the values of the angles of the triangle ABD when D moves.



The aim is that students observe that the angle in D is always 90°, and thus conjecture an important theorem in elementary geometry. But in this learning situation everything is given, and thus, despite appearances, there is no real place for mathematical inquiry. Note nevertheless that a small change would be sufficient to create a substantial difference. For instance, students could be given the segment [AB] and a point C that would be mobile in the plane, and asked to delimitate the region of the plane where point C could stand so that the angle in C of the triangle ABC is acute.

Unfortunately, research tends to show that technology is more often used in classrooms to support ostensive teaching practises (showing things to the students) or pseudo-experimental activities where students are led step by step along a worksheet, than to promote authentic inquiry-based approaches. Thus, it is important to pay particular attention to the resources built for supporting IBME through the use of technology.

Another issue arising when technology is used for conjecturing properties such as the one in Box 3 concerns the extent to which the experimental work supported by technology is likely to pave the way towards proof of the conjecture. Very often, both the problems selected and the way they are managed in the classroom tend to completely disconnect these two essential phases of the inquiry process. Educational research in mathematics shows that these two phases of mathematical inquiry can be treated in a connected manner (Hanna & De Villiers, 2012). Particular attention should also be paid to this aspect in the development of technological resources for IBME.

Dynamic geometry programmes and spreadsheets are the technologies most frequently used in mathematics education for promoting experimental practises. Mathematics curricula now require their use in many countries, at least in secondary education. Nevertheless, as pointed out in *Learning Through Inquiry*, nowadays technology has much more to offer for supporting IBME (i.e. through the use of sensors, simulation tools, applets and Internet resources). Many examples of this are provided by the educational literature (Hoyles & Lagrange, 2010). The Comenius project EdUmatics[8], for instance, aims at the collective development of resources for the professional development of European secondary mathematics teachers in the educational use of technology. One of the many resources developed within this project is based on a situation where students are asked to use a specific programme to test their reaction times and then to compare the different series obtained for the same student and for different students, and find the appropriate ways to make these comparisons. Such situations can be used both as starting points for developing the statistical portion of the mathematics curriculum in an inquiry-based perspective, and for establishing productive connections with natural sciences teaching.

1.5 IBME and other approaches in mathematics education

As pointed out in the introduction, even if the use of the expression IBME is rather recent, it should not be forgotten that IBME perspectives and values resonate with approaches that have existed and been developed in the field of mathematics education for decades. To mention just some of these:

- the problem-solving tradition going back to Polya (1945);
- the approaches initiated by Freudenthal in the Netherlands and Brousseau in France in the late sixties, which have progressively matured and are now known as Realistic Mathematics Education (RME) (Freudenthal, 1973), (Gravemeijer, 1999) and the Theory of Didactical Situations (TDS) (Brousseau, 1997);
- the modelling approaches, already mentioned;
- the socio-cultural approaches, especially those referring to the idea of community of practise (Lave and Wenger, 1991) or to Bakhtin's dialogic perspective (1981);
- anthropological approaches such as the Anthropological Theory of Didactics (ATD) initiated by Chevallard in the mid-eighties (Bosch & Gascón, 2006), or critical perspectives (Skovsmose, 1994).

To a certain extent, each of these constructions approaches mathematics education from an inquiry-based perspective and thus has something to offer to IBME, but each of them also tends to shape it in a particular way.

In the problem solving tradition, particular attention is paid to the development of problem solving skills and of associated metacognitive competences (Schoenfeld, 1992), which has often led to prefer the originality and challenging character of problems proposed to students over their contribution to an organised progression of mathematical ideas.

This contrasts with constructions such as RME and TDS which, despite their differences, share the ambition of structuring the whole educational enterprise through progressively solving epistemologically sound problems that make sense for the students.

8 www.edumatics.eu

In modelling approaches, connections between mathematics and the external world become the focus of attention. However, as is the case with problem solving approaches, tension can exist between modelling as a tool for developing connected mathematical knowledge and modelling as an object per se, when the development of modelling competences becomes an essential goal.

In socio-cultural approaches, particular attention is given to the social and cultural dimension of the inquiry process, to the ways it affects the communities involved (Jaworsky, 2004), and to dialogic interaction among students and between the students and the teacher (Wells, 1999). Within the Fibonacci Project, from this perspective, the dialogic approach developed around the work of Peter Gallin (Ruf & Gallin, 2005; Gallin, 2008) is an important source of inspiration.

In ATD, particular attention is paid to the institutional conditions and constraints that shape the inquiry process. The proposal is to rebuild mathematics education around an inquiry-based perspective through the innovative idea of "programme of study and research", inspired by Herbartian pedagogy (Chevallard, 2011).

In critical approaches, emphasis is put on the role that inquiry can play in questioning the way our societies function, and in promoting citizenship, social justice, and equity values.

We could develop this topic further, but what precedes suffices to show that IBME is an idea that is still under construction, and to which all these different perspectives have something to contribute. Each of these approaches highlights an important facet of IBME and provides both useful concepts and methods for approaching them, and resources and empirical studies that can inspire teaching and teacher education. Of course, organising these various approaches is a challenging enterprise: it implies building the appropriate bridges among them and the appropriate concepts to name them, as well as what is known within networking theory as a '*semiosphere*' where communication is made possible.

1.6 Final comments

Box 4 presents a hypothetical example of IBME in action, inspired by a workshop conducted in Bayreuth during a field visit in the framework of the Fibonacci Project and by experiences carried out in Italy (Martignone, 2010).

Box 4. Inquiry around pantographs

To start the inquiry process, students could be asked to manipulate different types of pantographs and to explore their respective behaviour. This would lead them to conjecture that this mechanical device allows the construction of enlarged representations of objects and drawings. Students could then be invited to use dynamic geometry to create virtual avatars of real pantographs, in order to facilitate a more systematic exploration of their possibilities. Dynamic geometry would thus support the connection between a technological system and a mathematical system in the modelling process. A study of these virtual artefacts would be then organised in the classroom, and the questions to be answered would be articulated and refined through dialogic interaction between the students and the teacher. The work would be distributed among groups of students acting as a community of inquiry.

This study would lead the students to encounter homothetic transformations which could be new for them, and motivate systematic studying of their properties, combining the use of dynamic geometry and paper and pencil work under the guidance of the teacher. At the end of the inquiry process, students would have understood why pantographs enlarge drawings but also made sense of the physical limitations of this technology. They would know how to make a pantograph capable of producing a given enlargement and how to combine the use of several pantographs for enriching their respective possibilities. They would have learned or consolidated their knowledge about homothetic transformations, about the kind of problems that can be solved through these transformations and about the role they play in geometry. They would probably also have learned about the history of these objects and the ingeniousness of those who created and progressively refined them. Some students could even consider building and commercialising pantographs for supporting community projects, as was the case in Italy (Martignone, 2010).

This inquiry project – which can also be considered a study and research programme from an ATD perspective – is an example of an activity in which the main ideas about IBME developed in this text would be implemented in an ideal manner. But there is no doubt that creating the conditions for making IBME viable is very difficult today in many educational systems. The structure of the curriculum, classroom organisation and mathematics schedules, teacher training, the prevalent didactical contract, and many other factors, create obstacles that are not easily overcome. As is the case in science, moving towards IBME is a long journey that generally begins with the implementation of much more modest forms of inquiry in which teachers should value their students' questions, take joint action with their students on the basis of students' questions and productions (Sensevy, 2011), create the conditions for students to make connections within mathematics and with the external world, and favour the cultivation of inquiry habits of mind.

The key features of inquiry pedagogy described in the booklet *Learning Through Inquiry* pave the way in that direction. It is expected that this progressive evolution towards IBME

> *"will improve students' mathematical understanding, which will result in their mathematical knowledge becoming more robust and functional in a diversity of contexts beyond that of the usual school tasks. It will help students develop mathematical and scientific curiosity and creativity as well as their potential for critical reflection, reasoning and analysis, and their autonomy as learners. It will also help them develop a more accurate vision of mathematics as a human enterprise, consider mathematics as a fundamental component of our cultural heritage, and appreciate the crucial role it plays in the development of our societies." (p. 8)*

2. Towards Teaching and Learning Inquiry-Based Mathematics

Peter BAPTIST
University of Bayreuth, Germany

2.1 Towards new teaching in maths - why?

Most people do not have an accurate picture of mathematics. They view mathematics as a set of formulas to be applied to a list of problems at the ends of textbook chapters. The reason for this misjudgement is the way maths is often taught. The traditional bureaucratic approach to maths at schools means prescribing content, presenting content, and measuring student acquisition of that content. This process has often failed in the past, is not working well now, and will fail in future. Teaching must not be limited to a teacher demonstrating a method of calculation and students subsequently repeating it without reflection. ("Here is the rule and the example, now practise with 20 homework problems.") Learning mathematics is an active, constructive, cumulative and goal-oriented process. This must also be perceptible for the students. Rather than showing facts or a clear, smooth path to a solution, the teacher guides students via well-designed problems through an adventure in mathematical discovery. To achieve this goal we have to bridge the gap between what mathematics is and how this subject often is taught at school.

> *"Give a man a fish and he eats for the day;*
> *Teach a man to fish and he eats forever."*

This truism captures perfectly the flaw in the traditional approach to maths education. A dominating teaching style gives students fish, it does not teach them how to fish. Our schools have focused far too long on the transfer of knowledge. Students remain passive while the teacher talks, explains, and works through a few examples, and then gets the students to do a number of similar problems. We need a fundamental re-orientation, we need a more progressive approach to maths. Students learn with greater depth of understanding and sustainability when they are actively engaged in the process of discovering concepts. School is more than a place for lecturing, school is a place for learning.

2.2 What is special about maths teaching in the Fibonacci Project?

First we have to realise: there is no single way of successfully teaching mathematics. Many paths lead to this goal, some of them quite different from one another. In the Fibonacci Project we have said good-bye to the teacher-centred dominating lecturing style in classroom. We prefer an inquiry-based approach to mathematics[9]. This kind of education does not present maths as a ready-built structure to appropriate. We do not start with formulas or rules, we get them at most at the end of the learning process. Learning is an individual process that can be suggested and influenced from outside, but real learning processes take place within each individual. The possibilities of an external control are known to be effective only in a limited way. If learning is to be successful, the students must be able to tie their own think-nets. The more connections they establish between elements of knowledge, i.e. the denser and tighter the nets are woven, the more flexible thinking becomes.

Students are then more likely to be able to apply their knowledge in changed or unfamiliar situations. Therefore we encourage students to use their own informal problem-solving strategies – at least initially – and then we have to guide their mathematical ideas and attempts towards more effective strategies and advanced understanding. What will take time and effort is learning how to think mathematically. That is what teachers should focus their time on.

There are certain basic fundamental concepts that are characteristic for our maths teaching at Fibonacci schools. Here are some examples:

- Less focus on passing factual knowledge to students, more focus on independent problem solving.
- Less focus on merely computing and manipulating formulas, more focus on understanding.
- Less focus on isolated problems, more focus on problems within a context.
- Not only focus on acquiring particular maths skills and results, but also focus on the necessary learning processes and strategies.

The implementation of these central concepts leads to an inquiry-based approach to mathematics. Inquiry-based mathematics education (IBME) will at least improve students' attitudes towards mathematics and their ability to use mathematics both in "real world" and inner mathematical contexts.

We must also consider the fact that receptiveness of many students to pure subject matter has substantially declined under the influence of everyday media consumption. This does not automatically mean that today's students are worse than those of former times; however they are different and thus they behave differently. So we do not need new content but rather a different approach to dealing with old content. We have to take this into consideration when it comes to teaching.

2.3 IBME – Stimulating acts

As a first step, inquiry-based mathematics education (IBME) implies that students have to be active. We introduce mathematics in the context of carefully chosen problems. In the process of trying to solve these problems the students gradually immerse into typical mathematical activities. According to Paul Halmos (1916 – 2006) the motto for our teaching and learning has to be: "Don't preach facts, stimulate acts."

9 See part 1 of this booklet and the Fibonacci Background Booklet *Learning Through Inquiry* (available in www.fibonacci-project.eu, in the *Resources* section) for details on the inquiry-based approach to mathematics education adopted in the Fibonacci Project.

IBME means the teacher is not an entertainer, the student is not just a consumer. We do not present ready-to-consume mathematics. Teachers must help students understand the concepts of mathematics, not just the mechanics of how to solve a certain problem. "Stimulating acts" stands for encouraging students to develop their own informal methods for doing mathematics. We ask them

- to question,
- to explore,
- to observe,
- to discover,
- to assume,
- to explain,
- to prove.

This list of practises shows basic activities for an inquiry-based approach to maths.

In mathematics, inquiry-based teaching and learning often starts with a certain problem or an experiment. Like in natural sciences, we may have hands-on experiments (e.g. paper knotting and paper folding, devices like a pantograph, games like tangram or tower of Hanoi), we have thought experiments (very typical for maths), and we have experiments on a computer screen with the help of appropriate software (dynamic worksheets).

But inquiry-based learning (IBL) goes beyond engaging students in activities. IBL places students in the role of researchers. The quality of students' investigations is linked to the quality of their own inquiries. The motivation for pursuing answers to their own questions is very strong.

What does IBL require from teachers and students? The following items have been adapted (slightly modified and shortened) from the teacher's edition of *Discovering Geometry* by Michael Serra.

- Develop skills for problem posing as well as for problem solving. Encourage the posing of new problems. You can foster problem posing by using question openers such as those listed in the next item.
- Develop a list of inquiry questions together with your students such as
  - What happens if…?
  - What if not…?
  - Why…?
  - How many…?
  - In general…?
  - What do we mean by…?
  - Is there a relationship…?
  - Under what conditions…?
  - What's the largest or smallest…?
  - What are the properties of…?
  - What other…?
  - How do you know…?
  - Is it always true that…?
  - Is it possible…?
  - How can you…?
  - Is there a similarity between…?
- Choose cooperative investigations as vehicles to promote activity and help students construct their own knowledge. The interaction, discussion, questions, suggestions, and ideas that students offer while working in their groups can benefit all group members. Some students will want more from the teacher than to help them help themselves. It may not be the way they have played the learning game, but it is the way a work environment runs. Students must take responsibility for their own learning and recognise that learning is an active, not a passive process.
- By grading take into account how well students are learning to do research in a team, not just the results of that research.

- Play the role of an experienced co-researcher rather than of someone with all answers. Don't give too many hints. Give encouragement for good thinking, not just for right answers. Treat right answers as discussion topics until the class agrees on them. As soon as you acknowledge a right answer, you often shut off thinking about the problem, even if students don't understand the answer. You will find that if you provide answers and explanations too quickly students may continue to expect and depend on your answers.
- Make clear to your students that mathematics is much more than a collection of facts and procedures. Many mathematical investigations don't have just one answer, and rarely is there only one valid approach to a solution. Justifying ideas and problem solving become more important than the actual solutions. The goal for students is to experience mathematics as a process of finding and connecting ideas. Let the students know that the thinking and problem solving skills they develop can serve them in all aspects of their lives.

Comparing Serra's IBL items with the basic patterns or key features of inquiry pedagogy adopted in the Fibonacci project[10] we recognise a strong similarity. This underlines the relevance of the basic patterns as an overarching structure for IBME.

2.4 The role of the key features of inquiry pedagogy in the Fibonacci Project

After the decision to shift towards new teaching and learning in maths, teachers need structuring elements to organise classroom work and learning processes. In the Fibonacci Project, these elements are called basic patterns or key features of inquiry pedagogy. Especially when starting the process of improving teaching and learning, teachers should concentrate on certain areas that are of specific interest. These may be such areas where teachers have diagnosed deficiencies and a lot of catching up to do. Thus the idea of the basic patterns was born – in analogy to the successful module concept we had developed for the German SINUS Project (1998-2007)[11].

Altogether we differentiate between nine basic patterns:

- Developing a problem-based culture
- Working in a scientific manner
- Learning from mistakes
- Securing basic knowledge
- Promoting cumulative learning
- Experiencing subject boundaries and interdisciplinary approaches
- Promoting the participation of girls and boys
- Promoting student cooperation
- Promoting autonomous learning

These basic patterns indicate which direction teaching should go. They are a unique distinctive feature of our project that clearly and positively differs from other IBL projects. The basic patterns are the underlying core structure both for teacher education and for classroom teaching. At the start of teaching – according to the Fibonacci philosophy – it is wise to choose one or two of these patterns. The specific choice depends on striking deficiencies in teaching and learning or on distinctive national features.

Especially in teaching mathematics the first basic pattern – developing a problem-based culture – plays an important role. Tasks and problems are starting points of mathematics education. Problems characterise mathematics lessons as part of an introduction to a new topic or as exercises at the end of textbook chapters. If we want to change teaching and learning maths, how we deal with problems in the classroom is essential.

10 See Fibonacci Background Booklet *Learning Through Inquiry*, available on www.fibonacci-project.eu, in the *Resources* section.
11 For detailed information on the German SINUS Project, see http://sinus.uni-bayreuth.de/2886/.

We have to differentiate several stages. At first we have three preparatory stages:

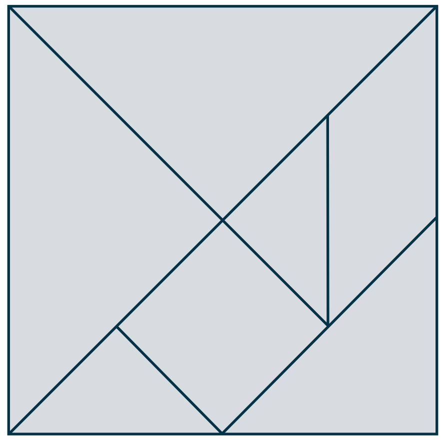
- *Exploring*: Students start learning by exploring texts, materials, situations and events.
- *Questioning*: Students ask questions to clarify an issue or pose a problem.
- *Collecting and Planning*: Students collect data and information. They think of a range of possibilities to answer open questions or to solve the problem.

A good theory is one side of a coin, but at school it is just as important to know appropriate examples for tackling problems. Here are some examples for classroom work:

- How much air does this hot-air balloon contain?

Foto: Kropsoq, commons.wikimedia.org

- A certain object has a volume 216 cm^3. Describe what it might look like.
- A tangram puzzle consists of seven geometrical shapes, so-called tans that are put together to form specific shapes. All seven pieces, which may not overlap, have to be used. Find different tasks and problems in connection with the tans.



The next stages in the process are

- *Deciding*: Students decide which possibility provides the best answer(s) to the questions or the solution to the problem.
- *Communicating*: Students choose the best way to present and explain their findings.
- *Looking back*: Students review their solution. Does it make sense? Is there a better way? They consider extensions and variations.

Our first basic pattern – developing a problem-based culture – aims at a larger variety of tasks and problems that allows individual and different solutions at various levels. We have to create problems that

- enable students to find different ways of solving problems,
- systematically repeat contents previously learnt,
- allow an open-ended approach,
- make use of a student's basic knowledge and connect it with newly acquired skills,
- can be transferred to various situations and contexts.

Furthermore we have to use a broad diversity of teaching methods and strategies when a new concept or phenomenon is introduced and elaborated, when well-known contents are applied to new situations, when the computer is used as a learning tool.

Stimulating and interesting problems and various ways for their solution are in the centre of the first basic pattern, developing a problem-based culture. Following the stages specified above, while tackling one of the given examples, we notice that more or less automatically other basic patterns are included, as for example:

- Working in a scientific manner
- Promoting student cooperation
- Promoting cumulative learning

Here we can see the relevance of basic patterns as structuring elements. With their help we are able to identify which areas are involved in the learning process. Thus teachers get an overview of which aspects they have taken into account. On the other hand, basic patterns show well-defined areas where IBME turns out to be useful.

2.5 Problem solving – the journey is the goal

In maths, an inquiry-based approach is often a problem-based approach. Therefore the first basic pattern plays a particularly important role for maths education, which is based on meaningful problems taken both from real situations and abstract mathematics. The six stages of a problem solving process described above are influenced by the grand master of successful problem solving George Polya (1887 – 1985). In his famous book *How to Solve It* he propagates the following approach:

- Understand the problem
- Devise a plan
- Carry out the plan
- Look back

Polya's and other comparable schemes are designed to provide guidance in working with problems. They facilitate access to systematic problem-solving approaches. In addition, they support the first steps from mere problem solving to getting more deeply involved in the context of the problem. Polya also has very clear ideas about the type of problems to be presented at school. He suggests problems that are about one-third geared to students' mathematical literacy and about two-thirds geared to their common sense.

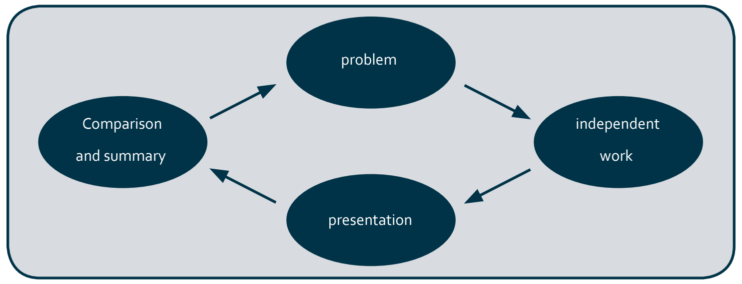
One cannot learn to solve problems just by obeying problem-solving rules. One can only learn by dealing with meaningful problems and analysing their solutions. It is crucial for success that such endeavours not be sporadic. This type of instruction must be given systematically.

What can you do if you are unable to find a promising approach to solve a current problem? Here, even Polya does not have an all-purpose formula up his sleeve. All he says is: "The first rule of discovery is to have brains and good luck", and "The second rule of discovery is to sit tight until you get a bright idea". A stock of strategies is certainly more helpful. An experienced problem solver gains these strategies by carefully analysing solutions. Such strategies are relevant not only for mathematics, but also for other subjects as well as for coping with problems in everyday situations. Here are some examples:

- testing and systematic re-testing by trial and error,
- working with special cases,
- generalising,
- finding analogies,
- restructuring,
- working forward and/or backward,
- going back to a known case,
- recognising patterns.

2.6 Problem-based teaching – Implementation in the classroom

If we want students to develop a living relationship to maths, they must not experience maths instruction as an artificial world that has nothing to do with their personal lives and feelings. This requirement grows from Hans Freudenthal's (1905 – 1990) idea of mathematics as a human activity. He felt that students should not be considered as passive recipients of ready-made mathematics, but rather that education should guide the students towards using opportunities to reinvent mathematics by allowing them to put it into practise.



Students must be given greater independence vis-à-vis the teachers. Students should learn to structure, process, and present their own ideas. Cognitive psychology stresses the enormous significance of autonomous learning. A successful learning process is active, constructive, cumulative, and goal-oriented. For school education, this means, among other things, that the teacher is not an entertainer, and the student is not a mere consumer.

For implementing these ideas in the classroom we can use the following structure for a teaching unit:

Discussion of the individual phases:

- The students should preferably begin with meaningful, challenging, and rich problems that can be solved in different ways and at different levels of formalisation.
- Students work independently over relatively long periods of time (as individuals and/or in small groups). The teacher monitors the students. At most, he acts as an advisor (helping them to help themselves). The teacher notes the approaches and/or solutions that are taking shape and then can select students for presentations.

- Selected students present their solutions and/or approaches to the problem. The various presentations are compared and discussed. The teacher should exercise restraint and only intervene when necessary. Such discussions foster the development of both daily and mathematical language.
- Following the students' active working phases, the teacher summarises the results. This teacher-centred phase can also be used to add further remarks and introduce new concepts and formalisms.

This approach to classroom teaching is based on interaction between students' active, independent working phases and teacher-centred instruction phases. Independent knowledge formation does not rule out instructional support or systematic transmission of knowledge. In the end, it is the interaction that makes for effective and sustainable learning processes.

In schools involved in the SINUS Project, this approach has been best practise. Comparing it with traditional methods of instruction, we found that

- learning subject matter is not limited to one single method but involves several processes handled by every student for him- or herself;
- students are prompted to work actively and independently;
- presentations of various solutions demonstrate different approaches and mathematical communication among students is promoted;
- initially the teacher is a mere observer and, at most, an advisor;
- the teacher does not offer explanations until the students have attempted to come up with a solution;
- in the students' active phase, weak and unmotivated students cannot duck for cover as effectively as they can in lecture-style teaching environments;
- initially, it is more time-consuming;
- at first it is unfamiliar, more complex, and more demanding both for students and teachers;
- this demanding type of instruction is also enjoyed by students.

Problem-based teaching and learning facilitates initial learning. But for promoting transfer, for deeper understanding and for sustainable learning we additionally have to study reasonable exercises and we have to build up a profound abstract knowledge by analysing and generalising the problems and exercises considered. This means recognising common structures, patterns, and strategies, using traditional symbolic notation, etc.

The main ingredients of a substantial learning environment can be summarised as

- problem-based teaching,
- meaningful exercises,
- linking new knowledge with prior knowledge.

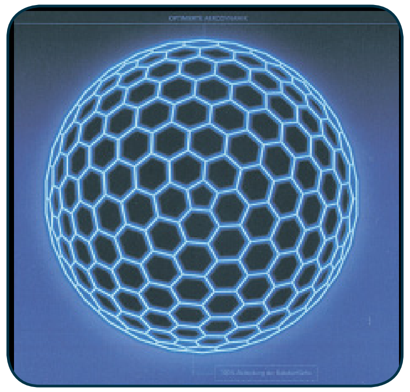
2.7 Stimulating acts – Surface of a golf ball

After having preached facts, now it's time to stimulate acts. The "golf ball" example demonstrates how we can initiate a pure geometrical topic with a real-life situation. Furthermore, students have the opportunity to develop their own problem solving strategies. The surface of a golf ball is – in contrast to a table tennis ball – not smooth but fairly uneven. Why do golf balls have these so-called dimples? Their shape and their arrangement differ from brand to brand. A lot of research work is done in this field. The reason for all these efforts: Players want their balls to go as long and straight as possible. Some years ago the Callaway Company announced a very innovative golf ball. The major message of an advertisement in a golfer's magazine was: "A revolutionary concept for golf ball surfaces: HEXAGONS," and the ad further detailed: "Our goal was to develop the most progressive and most aerodynamic golf ball in the world. Its patented structure of hexagons covers 100% of the surface of the ball. ..."

By hexagon Callaway understands a regular hexagon. Starting from a vertex we add further congruent regular hexagons. It turns out that on each vertex three hexagons exactly fit together (why?). By proceeding in all directions we get a complete covering of the plane without gaps or overlaps, a so-called tessellation.

Callaway Golf advertisement

Now we have a serious problem. How can we arch such a flat grid? How can we cover the spherical surface of a ball with such a grid? Is it a kind of magic by the Callaway engineers? The solution is fairly simple. The ad writer has ignored that the grid of the ball surface contains not only hexagons but also some regular pentagons. These pentagons cause the rounding. We know this kind of a structure from the surface of a traditional soccer ball.



Callaway golf advertisement

The ad in the golfer's magazine stimulates a number of further activities:

- Construction of regular polygons with ruler and compass
- Making regular pentagons and hexagons with the help of strip(s) of paper
- Tessellation of the plane with triangles or quadrangles
- Tessellation of the plane with regular polygons
- Investigation of Platonic and Archimedean solids
- Beauty and significance of symmetry

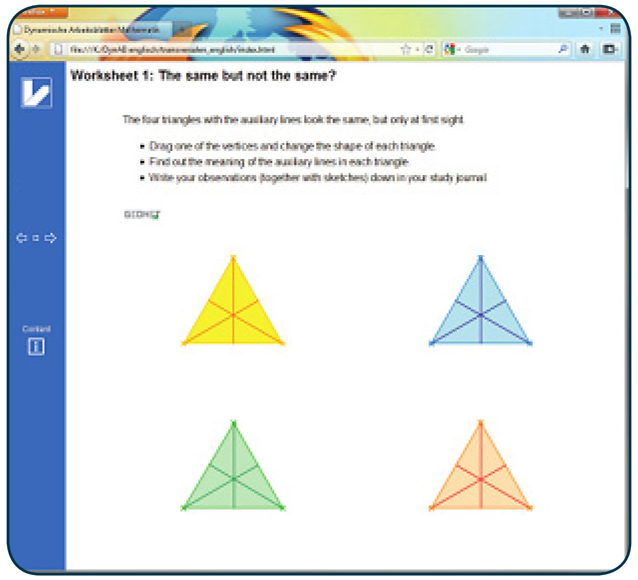
2.8 Experiments on the computer screen – Records in a study journal

Technology is no substitute for conceptual understanding, but it is useful for deepening that understanding. The use of software and technology helps students visualise and examine relationships in a dynamic environment. Again, here teaching takes place in a problem-oriented manner. We start with specific activities on the computer screen. The students are asked to do some experiments by themselves. The aim is to discover certain properties or relationships and to write down assumptions. Mathematics turns out to be an experimental science. The PC is the laboratory, a dynamic configuration on the screen represents the experiment.

To ensure sustainability, both the experiments and the results have to be recorded in a study journal. While exploring dynamic worksheets students often are asked to take notes. We prefer the old-fashioned writing and sketching with a pencil on paper. This method perfectly complements working with the electronic dynamic worksheets. The students use their study journal for

- sketching meaningful figures,
- describing their observations,
- phrasing assumptions,
- writing down proofs,
- expressing individual impressions and comments.

These individual records are used as a basis for an overview of the discussed subject matter. We can also offer so-called result-sheets as PDF-files. They contain a summary of the learning environment and complete the individual records of the students.



2.9 Design of a computer-based lesson

In a learning arrangement with dynamic worksheets the students have to go through learning processes in three stages:

- by themselves,
- together with other students,
- discussing with the whole class and/or the teacher.

After studying a dynamic worksheet for themselves the students are asked to exchange their ideas and results, to compare and to complete them. Here they practice orally and in writing how to express themselves mathematically. This active discussion leads to a deeper understanding of the mathematical topic. Working with dynamic worksheets shows that learning is not a passive process.

The teacher chairs the students' presentation and the discussion of the results. Afterwards – in a teacher-centred stage – he/she gives a summary and connects the new results to prior mathematical knowledge. Furthermore he/she may show standard proofs and introduces new mathematical terms, if necessary. Of course this lesson pattern is not dependent on using ICT (cf. the chapter above on problem-based learning).

Reasons for using dynamic worksheets can be summarised as follows:

- Students have to be active.
- Independent working is promoted.
- Students are largely able to decide their own learning speed.
- The experimental access arouses their interest.
- Moving and changing of configurations leads to new insights.
- Dynamic visualisation supports the understanding of geometrical properties.
- No previous technical knowledge is necessary.
- No familiarity with mathematical software is needed.

2.10 Guiding teaching concept – Reconsidering one's own teaching

There is no doubt that successful instruction has an individual face that is primarily that of the individual instructors. Well-prepared project ideas and materials provide inspiration, but implementation always has a personal touch.

What distinguishes successful mathematics education at school? How can IBME be implemented? Access to the "Fibonacci philosophy" is best achieved through conscious consideration of one's own teaching. Here certain central themes can serve as a means of orientation. These themes take five different aspects of teaching into consideration:

1. Teaching style
2. Work with problems/tasks
3. Technical content
4. Type of achievement testing
5. The role of mathematics teachers

Reflection based on these central themes also makes sense in pre-service teacher training. Although first-time instructors usually have only very limited teaching experience (if any), these central themes clearly point to crucial areas for subsequent instructional activity.

1. Reconsider your teaching style

- You do not represent the focus of instruction, your students do.
- Support your students in their learning; avoid lecturing them.
- Encourage your students to explore their own paths to learning.
- Provide suggestions and assistance towards self-help.
- Clearly separate learning and testing situations.
- Vary forms and methods of classroom teaching.
- In the final analysis it is not you but your students who are responsible for their learning progress.

2. Reconsider your work with problems

- Do not have your students simply produce answers, make them get involved with the respective questions. The journey is the reward.
- Make it possible to actively and productively work with problems:
  - initiate problems and tasks;
  - vary problems;
  - recognise patterns;
  - prepare strategies for solutions.
- Find different paths towards solutions and then take them.
- Link everyday knowledge and mathematical knowledge in a meaningful way.
- Have students keep study journals. Teach them how to express problem-solving and learning processes in written form.

3. Reconsider the subject matter

- Limit yourself to the fundamental content.
- Emphasise the essential ideas of the respective subject matter.
- Treat the content in an appropriate and interesting context.
- Attach value to discovering and working out relationships involving content and structure.
- Reduce the prevailing orientation towards calculation in favour of a focus on understanding.

4. Reconsider your conventional manner of testing

- Does testing always have to involve calculation?
- Can a test item take the form of a description?
- Can explanations and justifications be built into a "traditional problem"?
- Can problems be prepared in such a way that various solutions are possible and make sense?
- Variation of test items can be more demanding than formal application of a calculation method.
- Evaluate how study journals are maintained.
- Include checks of learning objectives in planning your teaching units.

5. Reconsider your own role as a mathematics teacher

- Express your enthusiasm for mathematics.
- Repeatedly emphasise the importance of mathematics, particularly in terms of culture, technology and industry.
- Show personal interest in the subject matter that you teach.
- Continue to be actively involved in mathematics: problem solving, competitions, popular science literature, etc.
- Avoid being a "lone wolf", rely on cooperation among the student body and/or teaching staff.
- Through your teaching show that mathematics is a vibrant, constantly developing discipline.

A first step is taken by reconsidering one's own teaching on the basis of the central themes indicated. Thus the foundation is laid for a change in teaching. What remains then is for implementation to take place in accordance with the motto cited from Paul Halmos: "Don't preach facts, stimulate acts." Thus we follow one of the key principles of IBME: that students should actively participate in the learning process. In a stimulating learning environment they should have the opportunity to build up their own knowledge and understanding.

References

- Aldon, G., Cahuet, P.Y., Durand-Guerrier, V. Front, M., Krieger, D., Mizony, M., Tardy, C. (2010). *Expérimenter des problèmes innovants en mathématiques à l'école*. Lyon: INRP.
- Artigue, M., Blomhøj, M. (2012). *Conceptualization of inquiry based education in mathematics*. Preprint.
- Bakhtin, M. (1981). *The dialogic imagination: Four essays by M.M. Bakhtin* (M. Holquist & C. Emerson transl.). Austin: University of Texas Press.
- Blomhøj, M. and T. Højgaard Jensen (2003): Developing mathematical modelling competence: Conceptual clarification and educational planning, *Teaching Mathematics and its applications* 22 (3), pp. 123-139.
- Brousseau, G. (1997). *Theory of didactical situations in mathematics.* Dordrecht, Kluwer.
- Brousseau G., Brousseau N. (1987) *Rationnels et décimaux dans la scolarité obligatoire*. I.R.E.M. de Bordeaux.
- Bosch M., Gascón J. (2006). 25 years of the Didactic Transposition, *Bulletin of the International Commission on Mathematical Instruction* 58, 51-65.
- Chevallard, Y. (2011). La notion d'ingénierie didactique, un concept à refonder. Questionnement et éléments de réponse à partir de la TAD. In Margolinas, C., Abboud-Blanchard, M., Bueno-Ravel, L., Douek, N., Fluckiger, A., Gibel, P., Vandebrouck F. & Wozniak F. (Eds.). *En amont et en aval des ingénieries didactiques*, pp. 81-108. Grenoble: La Pensée Sauvage.
- Dewey, J. (1938). *Logic: The theory of inquiry*. New York: Holt.
- Dewey, J. (1933). *How we think: A restatement of the relation of reflective thinking to the educative process*. Boston: Heath.
- Dewey, J. (1926). *Democracy and education*. New York: Macmillan.
- Douady, R. (1986). Jeux de cadres et dialectique outil-objet. *Recherches en Didactique des Mathématiques*, 7/2, 5-32.
- ERMEL (1999). *Vrai ? Faux ? ... On en débat !* Paris: INRP.
- Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht: Kluwer Academic Publishers.
- Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht: Kluwer Academic Publishers.
- Gallin, P. (2008). Den Unterricht dialogisch gestalten – neun Arbeitsweisen und einige Tipps. In: Ruf, U., Keller, S. & Winter, F. (Eds.): *Besser lernen im Dialog*. Seelze-Velber: Kallmeyer Verlag, 96 – 108.
- Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, 1, 155-177
- Halmos, P. (1975). *The Teaching of Problem Solving.* In: *The Problem of Learning to Teach*. The American Mathematical Monthly (Vol. 82, No. 5)
- Hanna, G., De Villiers, M. (Eds.) (2012). *Proof and proving in mathematics education. The 19th ICMI Study*. Springer.
- Hersant, M., Orange-Ravachol, D. (to appear). La démarche d'investigation, les mathématiques et les SVT : des problemes de demarcation aux raisons d'une union. *Proceedings of the Conference EMF 2012*, Geneva, February 2012.

- Hoyles, C. & Lagrange, J.B. (2010). *Mathematics Education and Technology – Rethinking the Terrain. The 17th ICMI Study*. Springer
- Jaworski, B. (2004). *Grappling with complexity: co-learning in inquiry communities in mathematics teaching development*. Plenary address at PME 28.
- Kaiser G., Blum, W., Borromeo Ferri, R., Stillman, G. (2011) (Eds.). *Trends in Teaching and Learning of Mathematical Modelling*. Dordrecht: Springer.
- Lakatos, I (1975). *Proofs and Refutations*. Cambridge University Press.
- Lave, J. & Wenger, E. (1991). *Situated Learning: Legitimate Peripheral Participation*. Cambridge, UK: Cambridge University Press.
- Martignone, F. (Ed.) (2010). *Scienze e Tecnologie in Emilia-Romagna- Azione 1- Laboratori delle Macchine Matematiche per l'Emilia Romagna*. Technodid Editrice.
- National Research Council (2000). *Inquiry and the national science education standards.* Washington, DC: National Academy Press.
- Polya G. (1945, 2004). *How to Solve it ?* Princeton University Press.
- Rocard M, Csermely P., Jorde D., Lenzen D., Walberg-Henriksson H. & Hemmo V. (2007). *Scientific education now: a renewed pedagogy for the future of Europe*. Commission Européenne, Direction générale de la recherche, Science, économie et société.
- Ruf, U. & Gallin, P. (2005): *Dialogisches Lernen in Sprache und Mathematik. Austausch unter Ungleichen. Grundzüge einer interaktiven und fächerübergreifenden Didaktik* (vol. 1) and *Spurenlegen – Spuren lesen. Unterricht mit Kernideen und Reisetagebüchern* (vol. 2). 3rd revised edition of the 1st edition 1998. Seelze-Velber: Kallmeyer.
- Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, meta-cognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for Research on Mathematics Teaching and Learning*. New York: MacMillan, 334-370.
- Sensevy, G. (2011). *Le sens du savoir. Eléments pour une théorie de l'action conjointe en didactique.* De Boeck Editeur.
- Serra, M. (2007). *Discovering geometry: an investigative approach*. Key Curriculum Press.
- Sierpinska, A. (1996). *Understanding in Mathematics*. Falmer Press.
- Wells, G. (1999). *Dialogic inquiry: Towards a sociocultural practice and theory of education.* Cambridge, UK: Cambridge University Press.

Consortium members

Credits

Template design : Brice Goineau

Graphic design : www.lezard-graphique.net

This document complements the Fibonacci Background Booklet on inquiry-based education in mathematics and science entitled Learning Through Inquiry. It is structured in two main parts. The first part, authored by Michèle Artigue, deepens the reflection initiated in that booklet on the nature of inquiry in mathematics education and how it relates to inquiry in science education. The second part, authored by Peter Baptist, focuses on the implementation of inquiry-based mathematics education and the important changes in educational practises that this move requires. The ideas developed in these two texts are illustrated by examples. The document especially refers to the nine key features of inquiry pedagogy (basic patterns) adopted in the Fibonacci Project.

Contact details

Fondation *La main à la pâte*
1 rue Maurice Arnoux
92120 Montrouge - FRANCE
+33 (0) 1 58 07 65 97
Fibonacci@fondation-lamap.org

WWW.FIBONACCI-PROJECT.EU